J. Fluid Mech. (1968), *vol.* 33, *part* 1, *pp.* 1–20
Printed in Great Britain

Viscoelastic properties of fine-grained incompressible turbulence

By S. C. CROW

Lawrence Radiation Laboratory, University of California, Livermore, California†

(Received 3 April 1967 and in revised form 2 January 1968)

A number of shear-flow phenomena can be explained qualitatively if turbulence is regarded as a continuous viscoelastic medium with respect to its action on a mean field. Conditions are sought under which the analogy is quantitative, and it is found that the turbulence must be fine-grained and the mean field weak. For geometrical convenience the turbulence is assumed to be nearly homogeneous and isotropic so that body forces are required to maintain it. The turbulence is found to respond initially to an arbitrary deformation as an elastic medium, in which Reynolds stress is linearly proportional to strain. Three processes that cause the resulting Reynolds stress to relax are distinguished: viscous diffusion, body-force agitation and non-linear scrambling. It is argued that, regardless of which process dominates, Reynolds stress evolves in a continuously changing mean field according to a viscoelastic constitutive law, relating stress to deformation history by means of a scalar memory function. The argument is carried through analytically for weak turbulence, in which non-linear scrambling is negligible, and the memory function is computed in terms of the wave-number-frequency spectrum of the background turbulence. In the course of the analysis, a new type of Reynolds stress arises related to the passage of the turbulence through its sustaining environment of body forces. It is found that the mean field must be surprisingly weak for this 'translation stress' to be negligible. Applications of the viscoelasticity theory of turbulent shear flow are discussed in which body forces and therefore translation stress are absent.

1. Introduction

One can hardly watch a body of turbulent fluid, a wake behind a ship for example, without imagining that the chaotic and apparently fine-grained eddies, like crystals of a metal or molecules of a gas, are elements of a state of nature, and that the body as a whole is made of turbulence and possesses mechanical properties intrinsic to its turbulent composition. Reynolds (1894) probably had some such description in mind when he defined turbulent stress by analogy with molecular momentum transport, but subsequent research has shown that there can be no comprehensive theory of Reynolds stress analogous to the kinetic theory of molecular viscosity. A gas is dominated at the atomic level by thermal

† Present address: National Physical Laboratory, England.

chaos and is only weakly perturbed by macroscopic forces. In the language of statistical mechanics, a gas is in contact with a 'thermal bath', an inexhaustible source of chaos that prevents macroscopic perturbations from accumulating. Turbulence, on the contrary, is maintained usually by a mean field rather than by random body forces, which would simulate a thermal bath. The turbulence must be strongly coupled to the mean field that drives it and cannot acquire intrinsic mechanical properties. In modern treatments of turbulent shear flow, the Reynolds stress appears as a purely formal consequence of ensemble-averaging the advection term in the Navier–Stokes equation.

The intensity of the coupling between mean and turbulent flow can be represented by two dimensionless quantities:

$$\epsilon = l/L,$$

the ratio of an eddy size l to a mean-field scale L, and

$$\alpha = A\theta,$$

the product of a mean rate of deformation A and a turbulent relaxation time θ. If $\epsilon \sim 1$, then, at any point, average turbulent properties depend on the whole mean flow rather than on local conditions alone. If $\alpha \sim 1$, then the mean flow has a strong effect on the shape of the eddies. Casual observation often suggests that $\epsilon, \alpha \ll 1$, in which case one could reasonably regard turbulence as a medium having intrinsic mechanical properties, in particular, a constitutive law relating Reynolds stress to deformation history. Careful measurements, however, always show that $\epsilon, \alpha \sim 1$ for values of A and L related to the mean field driving the turbulence.

In spite of the fundamental difficulty that $\epsilon, \alpha \sim 1$, some intrinsic mechanical attributes of turbulence have been recognized. The Prandtl mixing-length theory is based on the fact that turbulence, in a steady mean flow, diffuses momentum and offers viscous resistance to shear. The rapid-distortion theory of Taylor (1935), on the other hand, implies that the initial response of turbulence to a sudden deformation is elastic, not viscous (Taylor dealt only with irrotational deformation, but the conclusion is general, cf. § 3). The Prandtl and Taylor results are consistent under the assumption that turbulence responds elastically to a deformation imposed within a time much less than a turbulent stress-relaxation time θ, but offers viscous resistance if the time scale of the mean field is long compared with θ. Clauser (1956) used the concept of relaxation time to justify the two-layer model of the boundary layer: the wall layer relaxes quickly and achieves local equilibrium, but a packet of eddies convected through the outer layer retains information about the state of deformation far upstream of its current location. Viscous response, elastic response, and a stress-relaxation time to determine which takes place in any particular instance are the attributes of a viscoelastic fluid.

It has become apparent in the last decade that a viscoelastic constitutive law for turbulence might explain a number of seemingly unrelated phenomena. A mean cross-sectional flow develops when a turbulent fluid passes down a non-circular pipe. Rivlin (1957) noted that such a secondary flow implies the existence of normal Reynolds stresses similar to the stresses that would develop if the fluid were non-turbulent but also non-Newtonian. Liepmann (1961) proposed that a non-Newtonian constitutive law for turbulence might be found that would account for the large-eddy structure and self-excited oscillations of turbulent boundary layers. Townsend (1966) studied the instability of the bounding surfaces of free turbulent regions—jets and wakes—under the alternate assumptions that turbulence behaves as a viscous medium and as an elastic medium.

The purpose of this paper is to unite these various ideas, so far as possible, into a viscoelasticity theory of turbulent shear flow. The theory improves upon the speculations of Rivlin and Liepmann by incorporating analytical results for the initial elastic response of turbulence. It is necessary to assume that $\epsilon, \alpha \ll 1$ in order to obtain a viscoelastic constitutive law from the equations of motion, so the mean field inducing the Reynolds stress cannot be regarded as the source of the turbulence. The fine-grained turbulence could be maintained against molecular dissipation in various ways: by a small-scale mean field, in which case the turbulence would be highly anisotropic prior to its interaction with a superimposed large-scale mean field; by a flux of random vorticity from boundaries, in which case the turbulence would be inhomogeneous; by random body forces, a theoretically appealing, if physically unrealistic, source of turbulence. The third alternative is adopted here for the sake of geometrical simplicity, and the body forces are assumed to be statistically homogeneous and isotropic in space $\mathbf{x}$ and stationary in time t. In the absence of a mean field, the fine-grained turbulent medium is homogeneous, isotropic and stationary. The body forces themselves introduce a peculiar complication (translation stress), but the alternative sources of turbulence obscure with purely geometrical complications the fundamental processes of Reynolds-stress generation and relaxation. The flow is also assumed to be incompressible, although the results can be generalized easily for weakly compressible turbulence. The theory was developed originally to account for the damping of aerodynamic sound by the turbulence that generates it (Crow 1967*a*). Eddies of Mach number M emit sound of wavelength $L \sim M^{-1}l$, so $\epsilon \sim M \ll 1$ for low Mach number eddies. As an aerodynamically generated sound wave propagates outward from its parent eddy toward the boundary of a turbulent region, it decays as though it were propagating through a continuous viscoelastic medium. Such a situation might be realized in the interior of a star, where the turbulence actually would be driven by a random body force, namely buoyancy. The mathematical arguments set forth here have been reviewed elsewhere (Crow 1967*b*) in connexion with Burgers' one-dimensional model of turbulence.

In practice, random body forces seldom drive turbulence embedded in an incompressible shear flow. The source of turbulence ordinarily is either a small-scale mean field or a flux of random vorticity from an intensely turbulent region near a solid boundary. Both sources operate in a boundary layer (Townsend 1956, p. 235). The highly sheared mean field in the wall layer generates turbulence, which then diffuses outward under its own induction to form the outer layer. It is reasonable to hope that a turbulent boundary layer reacts viscoelastically to a secondary large-scale deformation, in other words, that the Reynolds stress *increment* induced by a large-scale deformation is related to the deformation by a viscoelastic constitutive law, generalized if necessary for an anisotropic turbulent medium. A viscoelastic constitutive law might be used to compute the Reynolds stress increment induced in a wind-driven boundary layer by an underlying ocean wave. Such stress increments may have a substantial effect on the rate of growth of ocean waves (Miles 1967). With respect to its effect on the boundary-layer profile, the turbulence cannot behave strictly as a viscoelastic fluid. But with respect to a long ocean wave underneath, the turbulent boundary layer as a whole might behave as a viscoelastic slab being driven over the wave crests. Turbulence in the water might also respond viscoelastically to the passage of the wave.

It is interesting to contrast the viscoelastic behaviour of turbulence in the limits $\alpha \to 0$ and $\epsilon \to 0$ with the results obtained by Pearson (1959) for the limits $\alpha \to \infty$ and $\epsilon \to 0$. Pearson showed that the kinetic energy density of turbulence grows without limit under an intense sustained strain, even in the presence of viscosity. In Pearson's limit $\alpha \to \infty$, the mean field dominates the eddy structure, and the turbulence is able to break loose from viscous decay, which is the only relaxation process that Pearson retains. In the opposite limit $\alpha \to 0$ studied here, structural changes in the turbulence subside before they can couple with the mean field to create a Pearson divergence.

2. Equations of motion

The velocity field can be resolved into an average $\mathbf{U}$ (the large-scale mean field) and a turbulent fluctuation $\mathbf{u}$. The kinematic pressure $P+p$ and body force $\mathbf{F}+\mathbf{f}$ can be resolved similarly. The fluctuations and the averages both vary in space and time. Because of the separation of length scales implied by the condition $\epsilon = l/L \ll 1$, the averages can be taken over an ensemble of flows, or over volumes V satisfying $l^3 \ll V \ll L^3$ in a particular flow. The equivalence of the two kinds of average permits the fine-grained turbulence to be treated as a continuous substance undergoing a particular motion $\mathbf{U}$. Brackets in the following equations denote ensemble or spatial averages.

Define a mean rate-of-deformation tensor

$$A_{ij} = \partial U_i / \partial x_j$$

and a Reynolds stress

$$\tau_{ij} = \tfrac{2}{3} T \delta_{ij} - \langle u_i u_j \rangle,$$

where T is the turbulent energy density $\langle \mathbf{u}^2 \rangle / 2$ in the absence of a mean field and is constant in space and time. The assumption that $\alpha \ll 1$ implies that $\tau_{ij} \ll T$, since the mean field, which is the exclusive source of anisotropy, can have only a small effect on the turbulence. Define D/Dt as a total time derivative convected in the *mean field only*:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j}.$$

Then the mean momentum and continuity equations are

$$\frac{DU_i}{Dt} + \frac{\partial P}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_j} + \nu \nabla^2 U_i + F_i, \tag{2.1}$$

$$A_{ii} = 0, \tag{2.2}$$

where ν is the kinematic viscosity. The fluctuation equations are

$$\frac{Du_i}{Dt}+A_{ik}u_k+\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}(u_i u_k+\tau_{ik}) = \nu\nabla^2 u_i+f_i, \tag{2.3}$$

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2.4}$$

It is convenient to assume that the random field $\mathbf{f}$ is solenoidal so that $\nabla.\mathbf{f} = 0$. In any case, $\mathbf{f} = \nabla\times\boldsymbol{\psi}+\nabla\phi$, and the potential ϕ can be absorbed in the pressure. Doing so at this stage eliminates some manipulation with projection operators in §§ 7–9. The fluctuation pressure can be written

$$\left.\begin{aligned} p &= p^T+p^A,\\ \text{where}\qquad \nabla^2 p^T &= -\frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j+\tau_{ij}),\\ \text{and}\qquad \nabla^2 p^A &= -2A_{ij}\frac{\partial u_j}{\partial x_i}. \end{aligned}\right\} \tag{2.5}$$

According to (2.2) and (2.3), the sum p^T+p^A preserves the incompressibility condition (2.4). p^T is the component of pressure due to convective acceleration of turbulence in its own field, and p^A is the component due to interaction between mean and turbulent fields.

An equation for the Reynolds stress is obtained by multiplying (2.3) by u_j, forming a new equation by transposing i and j, summing the results, and taking an average:

$$\frac{D\tau_{ij}}{Dt} = \tfrac{2}{3}T(A_{ij}+A_{ji})+\left\langle u_j\frac{\partial p^A}{\partial x_i}+u_i\frac{\partial p^A}{\partial x_j}\right\rangle-(A_{ik}\tau_{jk}+A_{jk}\tau_{ik})$$
$$+\langle\text{triple-interaction}\rangle+\langle\text{viscous}\rangle+\langle\text{body-force}\rangle. \tag{2.6}$$

The term involving p^A has been isolated; a similar term involving p^T is included implicitly as a triple-interaction term. The first two terms on the right of (2.6) represent stress generation by vortex stretching, as is made apparent in §3. The next term represents action of the mean field on an existing stress pattern, and the last three terms, which have been written symbolically, represent stress relaxation by non-linear scrambling, by viscous diffusion, and by body-force agitation respectively. They do not involve the mean flow explicitly, and they must sum to zero in the absence of a mean field, once the turbulence has attained equilibrium.

3. Initial elastic response

Suppose that prior to time zero the mean field is zero and the turbulence is resting in statistical equilibrium. The Reynolds stress τ_{ij} is zero, and the last three terms in (2.6) sum to zero. At time zero, body impulses instantaneously generate a mean field. No random impulses are applied, so the turbulent velocity field $\mathbf{u}$ suffers no impulsive change. The only fluctuation quantity that changes instantaneously is p^A, which, according to the last of (2.5), jumps from zero to a value

$$p^A(\mathbf{x}) = \frac{1}{2\pi}\int A_{mn}(\mathbf{x}+\mathbf{r})\,\frac{\partial u_n}{\partial r_m}(\mathbf{x}+\mathbf{r})\,\frac{d\mathbf{r}}{r},$$

where r is the magnitude of $\mathbf{r}$, and the integration extends over all space. The solution for p^A is valid only if the integral converges, but it must converge with probability one for a reasonable ensemble of flows. Immediately after the mean field is applied, therefore,

$$\left\langle u_j\frac{\partial p^A}{\partial x_i}\right\rangle = \frac{1}{2\pi}\int\frac{\partial}{\partial r_i}\left[A_{mn}(\mathbf{x}+\mathbf{r})\frac{\partial}{\partial r_m}R_{jn}(\mathbf{r})\right]\frac{d\mathbf{r}}{r}, \tag{3.1}$$

where $$R_{jn}(\mathbf{r}) = \langle u_j(\mathbf{x})\,u_n(\mathbf{x}+\mathbf{r})\rangle.$$

R_{jn} is independent of $\mathbf{x}$, since it is the correlation tensor of $\mathbf{u}$ evaluated at time zero, when the turbulence is homogeneous. After a short time Dt, a stress increment

$$D\tau_{ij} = \left[\tfrac{2}{3}T(A_{ij}+A_{ji}) + \left\langle u_j\frac{\partial p^A}{\partial x_i} + u_i\frac{\partial p^A}{\partial x_j}\right\rangle\right]Dt \tag{3.2}$$

has been generated. Equation (3.1) can be used to evaluate the pressure-interaction term in (3.2). The interaction pressure p^A couples $D\tau_{ij}(x)$ to the whole field $A_{mn}(\mathbf{x}+\mathbf{r})$ so that, in general, the Reynolds stress is a complicated non-local functional of the deformation field.

Let us now introduce the assumption that $\epsilon = l/L \ll 1$. L is a distance over which $A_{ij}(\mathbf{x})$ changes significantly, and l is a length such that $R_{ij}(\mathbf{r})$ is small for $r > l$. Then A_{mn} can be taken outside the integral in (3.1):

$$\left\langle u_j\frac{\partial p^A}{\partial x_i}\right\rangle = A_{mn}(\mathbf{x})\left[\frac{1}{2\pi}\int\frac{\partial^2}{\partial r_i\,\partial r_m}R_{jn}(\mathbf{r})\,\frac{d\mathbf{r}}{r}\right],$$

which is a linear function of the *local* value of the deformation rate. It is convenient to express R_{jn} in terms of its spectrum tensor Φ_{jn}:

$$R_{jn}(\mathbf{r}) = \int \Phi_{jn}(\boldsymbol{\kappa})\,e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\,d\boldsymbol{\kappa}.$$

After an elementary integration over $\mathbf{r}$,

$$\left\langle u_j\frac{\partial p^A}{\partial x_i} + u_i\frac{\partial p^A}{\partial x_j}\right\rangle = -2A_{mn}(\mathbf{x})\int\left[\frac{\kappa_i\kappa_m}{\kappa^2}\Phi_{jn}(\boldsymbol{\kappa}) + \frac{\kappa_j\kappa_m}{\kappa^2}\Phi_{in}(\boldsymbol{\kappa})\right]d\boldsymbol{\kappa}, \tag{3.3}$$

an expression for the pressure-interaction term in (3.2) valid for homogeneous and fine-grained but otherwise arbitrary turbulence. Since the turbulent background is supposed to be isotropic as well, the spectrum tensor has the form

$$\Phi_{jn}(\boldsymbol{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^2}\left(\delta_{jn} - \frac{\kappa_j\kappa_n}{\kappa^2}\right), \tag{3.4}$$

where $E(\kappa)$ is normalized so that its integral from $\kappa = 0$ to ∞ equals the total energy density T (Batchelor 1953, p. 49). The relations

$$\left.\begin{aligned} \frac{1}{4\pi}\oint n_i n_j\,dl &= \tfrac{1}{3}\delta_{ij}, \\ \frac{1}{4\pi}\oint n_i n_j n_m n_n\,dl &= \tfrac{1}{15}(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \end{aligned}\right\} \tag{3.5}$$

can be used in (3.3) to integrate products of the unit vector $\mathbf{n} = \boldsymbol{\kappa}/\kappa$ over all solid angles Ω. For fine-grained, isotropic turbulence,

$$\left\langle u_j \frac{\partial p^A}{\partial x_i} + u_i \frac{\partial p^A}{\partial x_j} \right\rangle = -\tfrac{2}{5}T(A_{ij} + A_{ji} - \tfrac{2}{3}A_{mm}\delta_{ij}). \qquad (3.6)$$

Since the mean field is incompressible, the term involving A_{mm} is zero. Equations (3.2) and (3.6) imply that a net stress increment

$$D\tau_{ij} = \tfrac{4}{15}T(A_{ij} + A_{ji})\,Dt \qquad (3.7)$$

has been generated by time Dt, the same stress that would have been generated in an elastic, isotropic medium having a shear modulus $\frac{4}{15}T$.

Some parts of the preceding argument depend on the assumption that the turbulence is three-dimensional. The argument can be recast for two-dimensional flow, and the result is that the analogue of (3.7) in two dimensions is identically zero. The difference arises because the vortex stretching mechanism is absent in two dimensions. In three dimensions, Reynolds stress is generated by systematically stretching random vorticity.

Suppose that a single column vortex stands vertically between two horizontal plates whose diameters exceed the core-diameter of the vortex. The fluid between the plates has density ρ, the vortex is assumed to be cylindrically symmetric, and the circulation around the vortex outside its core is Γ. If the plates are pulled farther apart, then the pressure drops inside the core of the stretched vortex, but not in the surrounding potential flow. The pressure drop tends to suck the plates back together. It is easy to show that if the separation between the plates increases by a fraction $A\,Dt$, then the suction force increases by an amount $Df = \rho\Gamma^2 A\,Dt/8\pi$, regardless of how the vorticity is distributed in the core. Df is independent of the sign of Γ and is proportional to the strain $A\,Dt$. The vortex therefore behaves like a Hooke's-law spring. It is satisfying, if not quantitatively useful, to visualize turbulence as a tangle of vortex springs. When the turbulence is merely rotated ($A_{ij} + A_{ji} = 0$, $A_{ij} - A_{ji} \neq 0$), no tension is generated, but when the turbulence is strained ($A_{ij} + A_{ji} \neq 0$), then the vortex springs aligned with a principal axis of positive strain stretch, those aligned with a principal axis of negative strain slacken, and the net effect is that the deformation is resisted as though the tangle were a continuous elastic medium.

4. Stress relaxation

Suppose now that the mean field is decelerated to zero at time Dt. The only terms left on the right-hand side of (2.6) are the last three, whose sum has acquired a small value by time Dt due to the systematic deformation $A_{ij}\,Dt$ of the eddies. As time passes, the Reynolds stress decays under the influence of the three relaxation processes which the terms represent. At high Reynolds numbers, for example, the stress relaxes as the turbulent field scrambles partially aligned vortex springs (tendency toward isotropy). It is not possible to treat non-linear scrambling analytically, but it is possible to assert that if the stress continues to depend linearly on the strain imposed between times 0 and Dt, then

$$D\tau_{ij}(t) = \tfrac{4}{15}T(A_{ij} + A_{ji})\,Dt\,\mathfrak{M}(t). \qquad (4.1)$$

That statement follows from the isotropy of the background turbulence: since there is no inherent preferred direction, the principal axes of stress continue to be aligned with the principal axes of strain. $\mathfrak{M}(t)$ is some dimensionless scalar. According to equation (3.7), $\mathfrak{M}(0) = 1$. If the turbulence was stable to begin with, then the stress must eventually relax back to zero, so $\mathfrak{M}(\infty) = 0$. If the turbulence is stable under the action of a steady but very weak mean field, then the integral relaxation time

$$\theta = \int_0^\infty \mathfrak{M}(t)\,dt \tag{4.2}$$

must be finite. Here the turbulence is assumed to be stable in that sense.

Equation (4.1) is valid for some $\mathfrak{M}(t)$ whenever the turbulence is isotropic and $\epsilon \ll 1$. Let us now assume further that $\alpha = A\theta \ll 1$ for a typical component A of a continuously changing deformation field $A_{ij}(\mathbf{x}, t)$. Then the mean field has only a small effect on the turbulence, and it is reasonable to assume that the mean field does not interfere with the processes that cause stress to relax, in other words, that $\mathfrak{M}(t)$ is independent of A_{ij}. Stress increments like (4.1) can therefore be summed for a continuously changing mean field to establish a constitutive law for incompressible, isotropic turbulence:

$$\tau_{ij} = \tfrac{4}{15}T\int_{-\infty}^{t} \mathfrak{M}(t-s)\,[A_{ij}(s) + A_{ji}(s)]\,Ds. \tag{4.3}$$

Equation (4.3) relates the stress sustained by a packet of turbulence (a volume V such that $l^3 \ll V \ll L^3$) to its strain history. 'Ds' means that the integration follows the packet through the mean field. Since $\alpha \ll 1$, the convected integral can be replaced with an integral at a fixed point in space, except in cases where the turbulence is subjected to a uniform translation as well as a weak deformation.

Equation (4.3) is the constitutive law of a linear viscoelastic material. $\mathfrak{M}(t)$ is a memory function for Reynolds stress and is supposed to depend solely on the nature of the background turbulence. In so far as (4.3) is correct, a shear-flow problem reduces to predicting $\mathfrak{M}(t)$ in terms of properties of the random forces $\mathbf{f}$.

Because (4.3) is linear in the deformation field, it cannot account for the normal Reynolds stresses observed in turbulent flow through non-circular pipes (Rivlin 1957). The normal Reynolds stresses depend on $O(\alpha)$ non-linearities, which have been neglected under the assumption that $\alpha \ll 1$. There is, however, an easy way to generalize (4.3) so that it includes one $O(\alpha)$ non-linearity, namely the action, represented by the third term in (2.6), of the mean field on an existing stress pattern. Instead of (4.3), try a constitutive law of the form

$$\tau_{ij} = \tfrac{4}{15}T\int_{-\infty}^{t} \mathfrak{M}(t-s)\,S_{im}(t,s)\,S_{jn}(t,s)\,[A_{mn}(s) + A_{nm}(s)]\,Ds. \tag{4.4}$$

A total time derivative of (4.4) reproduces the first two terms on the right-hand side of (2.6) to $O(1)$ [equation (3.4) may not be valid to $O(\alpha)$] if only

$$S_{ij}(s,s) = \delta_{ij}, \quad \text{all } s, \tag{4.5}$$

and reproduces the third term as well if

$$\frac{DS_{ij}(t,s)}{Dt} = -A_{ik}(t)\,S_{kj}(t,s). \tag{4.6}$$

If the deformation rate $A_{ij}(t)$ encountered by a packet of turbulence is given, then (4.6) can be solved in conjunction with (4.5) as an initial-value problem. It must be emphasized that (4.4) may not be quantitatively correct to $O(\alpha)$. It takes no account of changes in the pressure-interaction tensor (3.6) caused by $O(\alpha)$ departures of the spectrum tensor Φ_{jn} from its isotropic form (3.4), and no account of any $O(\alpha)$ interference between mean deformation and stress relaxation. As a non-linear viscoelastic constitutive law, however, (4.4) does give rise to normal stresses of a kind qualitatively consistent with those observed in turbulent shear flows (cf. §5).

5. Implications of a viscoelastic constitutive law

A viscoelastic constitutive law admits the possibility of purely elastic or purely viscous behaviour, depending on the size of a typical frequency, Ω say, of the deformation that a packet of turbulence experiences as it is convected through the mean field. Equation (4.3) approaches an elastic limit as $\Omega\theta \to \infty$ and a viscous limit as $\Omega\theta \to 0$.

Suppose first that $\Omega\theta \gg 1$, a situation compatible with the condition $A\theta \ll 1$ only if the turbulence undergoes very low amplitude, high-pitched oscillations. A passing water wave might impress such oscillations on the atmospheric boundary layer above the water or on the oceanic turbulence underneath. Choose co-ordinates moving with any over-all translation of the turbulence so that a packet of turbulence makes a small, oscillatory excursion $\mathbf{D}$ about its original position. Then $A_{ij} = \partial^2 D_i/\partial x_j \partial t$, and the integral in (4.3) can be evaluated at points fixed in space. It is easy to show that

$$\tau_{ij} = G_e \left(\frac{\partial D_i}{\partial x_j} + \frac{\partial D_j}{\partial x_i}\right), \tag{5.1}$$

where G_e is a turbulent shear modulus:

$$G_e = \tfrac{4}{15} T.$$

The proof depends on the fact that $\mathfrak{M}(t)$, in the limit $\Omega\theta \to \infty$, drops very slowly from its value $\mathfrak{M}(0) = 1$ compared with the rapid and self-cancelling oscillations of $\mathbf{D}$. One way of carrying out the proof is to define $\mathfrak{M}(t-s)$ as zero for negative values of its argument, extend the upper limit of the integral in (4.3) to ∞, apply the Fourier convolution theorem, and observe that the Fourier transform of $\mathbf{D}$ overlaps with only the tail of the Fourier transform of $\mathfrak{M}$; the tail depends on the jump of $\mathfrak{M}(t-s)$ from 0 to 1 across $t-s = 0$, not on the subsequent relaxation profile. Equation (5.1) is the constitutive law of an incompressible, elastic medium. If molecular viscosity is neglected, then the equations of motion (2.1) and (2.2) take the forms

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} - G_e \nabla^2 \mathbf{D} + \nabla P = \mathbf{F},$$

$$\nabla . \mathbf{D} = 0,$$

so the turbulence can support transverse shear waves propagating at a speed $c = (G_e)^{\frac{1}{2}}$. Moffatt (1965) has already called attention to that fact.

Consider next the opposite case $\Omega\theta \ll 1$. $\mathfrak{M}(t-s)$ drops to zero in (4.3) before the deformation rate $A_{ij}(s)$ has time to depart from its value at $s = t$, so

$$\tau_{ij} = \nu_e(A_{ij}+A_{ji}), \tag{5.2}$$

where ν_e is an eddy viscosity [cf. equation (4.2)]:

$$\nu_e = \tfrac{4}{15}T\theta.$$

Turbulence passing through nearly relaxed states behaves as an incompressible Newtonian fluid, provided that $\alpha \ll 1$, with an eddy viscosity ν_e proportional to the integral relaxation time θ. Viscous behaviour is a natural consequence of a relaxation process. Water, for example, ordinarily is considered to be a viscous fluid, but it is composed of loosly defined crystals held together by the same forces that bind crystals of ice. Water crystals and ice crystals both accept stress elastically. The difference is that stressed ice crystals are locked rigidly in place, whereas water crystals are knocked around and broken up quickly by thermal agitation. Ice has a very long relaxation time and is elastic; water forgets quickly and is viscous.

Suppose that $A\theta$ and $\Omega\theta$ are comparable, are fairly small, but are not negligible. Equation (4.4), in so far as it is valid, describes the resulting non-Newtonian contributions to the Reynolds stress. The integral can be evaluated to any desired order in α by expanding $A_{mn}(s)$ and $S_{im}(t,s)$ as Taylor series in powers of $(t-s)$. To order α^2,

$$\tau_{ij} = \nu_e\mathscr{T}_{ijmn}\{A_{mn}+A_{nm}\}, \tag{5.3}$$

where $\mathscr{T}_{ijmn}$ is an operator,

$$\mathscr{T}_{ijmn} = \delta_{im}\delta_{jn} - k\theta(\delta_{im}A_{jn}+\delta_{jn}A_{im}+\delta_{im}\delta_{jn}[D/Dt]),$$

involving the dimensionless constant

$$k = \frac{1}{\theta^2}\int_0^\infty t\mathfrak{M}(t)\,dt.$$

The assumption that k is finite imposes a stronger restriction on the stability of the turbulence than does (4.2). As an example of the use of (5.3), consider a steady, parallel shear flow in which the only non-zero component of $A_{ij}(\mathbf{x})$ is $A_{12}(x_2)$. According to (5.3), the shear stress $\tau_{12} = \tau_{21} = \nu_e A_{12}$, as in Newtonian turbulence, but the normal stress $\tau_{11} = -2\nu_e k\theta A_{12}^2$; all other components of stress are zero. Those results happen to be exact consequences of (4.4), true to all orders in α. The mean-square longitudinal fluctuation $\langle u_1^2\rangle$ therefore grows with A_{12}^2, and the mean-square transverse fluctuations $\langle u_2^2\rangle$ and $\langle u_3^2\rangle$ retain their equilibrium values. The conclusion is qualitatively consistent with observations of the high-shear region of boundary layers, where longitudinal fluctuations greatly exceed transverse fluctuations (Townsend 1956, p. 254).

6 Linear relaxation in weak turbulence

The analysis of §3 establishes the value of the memory function $\mathfrak{M}(t)$ at one time only, namely $\mathfrak{M}(0) = 1$. The subsequent behaviour of $\mathfrak{M}(t)$ depends on the three relaxation terms written symbolically in (2.6), the most significant of which, the triple-interaction term, is related to the mathematically intractable process of vortex scrambling. Relaxation by viscous diffusion and by body-force agitation *are* tractable, however, and $\mathfrak{M}(t)$ can be computed in the unlikely circumstance that those linear relaxation processes are dominant. It is instructive to check the assumptions underlying (4.3) by performing the computation.

The triple-interaction, viscous-diffusion, and body-force terms in (2.6) are of orders u^3/l, $\nu u^2/l^2$ and uf, respectively. Relaxation by non-linear scrambling is therefore negligible if either $u \ll \nu/l$ or $u^2 \ll lf$, which of course are the same conditions that permit the non-linear advection terms to be dropped from the fluctuation momentum equation (2.3). Since the non-linear advection terms are negligible, the body force f must balance either viscous damping, in which case $f \sim \nu u/l^2$, or acceleration, in which case $f \sim \omega u$, where ω is a typical frequency of the turbulent motion. The conditions for non-linear scrambling to be negligible are therefore $u \ll \nu/l$ or $u \ll \omega l$. The former corresponds to heavily damped turbulence and the latter to turbulence driven at a very high frequency, much higher than the frequency u/l associated with non-linear advection. The two inequalities represent two possible types of weak turbulence.

Suppose that equations (2.3)–(2.5) have been linearized with respect to terms quadratic in $\mathbf{u}$. Let us single for study the packet of turbulence that was centred at $\mathbf{x} = 0$ in the distant past. By time t the packet is centred at some new location $\mathbf{x} = \mathbf{D}(t)$. It is convenient to transform the linearized equations to a co-ordinate system

$$\boldsymbol{\xi} = \mathbf{x} - \mathbf{D}(t)$$

fixed to the centre of the packet and moving without rotation. In the vicinity of the packet, the mean velocity $\mathbf{U}$ can be expanded as a Taylor series in $\boldsymbol{\xi}$:

$$U_i = dD_i(t)/dt + A_{ij}(t)\,\xi_j + B_{ijk}(t)\,\xi_j\,\xi_k + \dots,$$

where $dD_i(t)/dt$ is the velocity at the centre of the packet, $A_{ij}(t)$ is the local deformation rate, and B_{ijk}, etc., are higher-order rates of distortion. If $\epsilon \ll 1$, the turbulence feels a uniform strain rate $A_{ij}(t)$, higher-order interactions with $B_{ijk}(t)$, etc., being inconsequential. Under the assumption that $\epsilon \ll 1$, the linearized momentum equation in moving co-ordinates is

$$\frac{\partial u_i}{\partial t} + A_{jk}(t)\,\xi_k\frac{\partial u_i}{\partial \xi_j} + A_{ij}(t)\,u_j + \frac{\partial p}{\partial \xi_i} = \nu\nabla^2 u_i + f_i[\boldsymbol{\xi} + \mathbf{D}(t), t], \tag{6.1}$$

where $\mathbf{u}$ is now a function of $\boldsymbol{\xi}$ and t, and the Laplacian involves derivatives with respect to $\boldsymbol{\xi}$. The linearized version of (2.5),

$$\nabla^2 p = -2A_{ij}(t)\frac{\partial u_j}{\partial \xi_i}, \tag{6.2}$$

maintains incompressibility if the body forces are solenoidal. Equations (6.1) and (6.2) are mutually consistent approximations of (2.3) and (2.5). They can be derived more rigorously, in the limit $\epsilon \to 0$, as the zeroth-order terms in co-ordinate expansions of the linearized versions of equations (2.3) and (2.5) (cf. Crow 1967*b*). Notice that no assumption has been made so far about the size of α.

The dependence of $\mathbf{f}$ on $\mathbf{x} = \boldsymbol{\xi}+\mathbf{D}$ has been retained in (6.1), because the statistical properties of $\mathbf{f}$ are most naturally referred to a fixed co-ordinate system. If the random forces were to drift with the mean field (that is, if the mean field were to induce a statistically significant dispersion relation between the frequency and wave-number of the space-time spectrum of $\mathbf{f}$), then the force field itself would have a memory, and the constitutive law for Reynolds stress would reflect that complication. In some situations the forces *would* be convected: $\mathbf{f}$ might be a buoyant force due to random heating in the fluid, for example. The analysis can be modified easily for such cases.

Whenever stress relaxation by non-linear scrambling is negligible, (6.1) and (6.2) can be used to find the Reynolds stress analytically. The stress can depend only on $A_{ij}(t)$ and $D_i(t)$ under the present assumptions. In so far as the Reynolds stress depends on $D_i(t)$, the viscoelasticity model (4.3) is inadequate.

7. Fourier analysis of the linearized problem

The velocity and body-force fields can be written in the form

$$\mathbf{u}(\boldsymbol{\xi},t) = \int \mathbf{v}(\boldsymbol{\kappa},t)\, e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\kappa},$$

$$\mathbf{f}(\mathbf{x},t) = \int \mathbf{g}(\boldsymbol{\kappa},t)\, e^{i\boldsymbol{\kappa}\cdot\mathbf{x}}\, d\boldsymbol{\kappa},$$

where the integrations are carried out over all wave-numbers $\boldsymbol{\kappa}$. The Fourier space transforms $\mathbf{v}(\boldsymbol{\kappa},t)$ and $\mathbf{g}(\boldsymbol{\kappa},t)$ are to be understood as generalized functions, representing inverse transforms like

$$\mathbf{v}_V(\boldsymbol{\kappa},t) = \frac{1}{(2\pi)^3}\int_V \mathbf{u}(\boldsymbol{\xi},t)\, e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}$$

evaluated over arbitrarily large volumes V. The Fourier transformation of (6.1) with respect to $\boldsymbol{\xi}$ is

$$\frac{\partial v_i}{\partial t} - A_{mj}(t)\,\kappa_m\frac{\partial v_i}{\partial \kappa_j} + \left(\delta_{im} - 2\,\frac{\kappa_i\kappa_m}{\kappa^2}\right) A_{mj}(t)\, v_j + \nu\kappa^2 v_i = e^{i\boldsymbol{\kappa}\cdot\mathbf{D}} g_i, \tag{7.1}$$

essentially a wave equation in $\boldsymbol{\kappa}, t$ space, with characteristic curves defined by $d\kappa_j/dt = -A_{mj}(t)\kappa_m$. The pressure equation (6.2) has been used to eliminate the Fourier transform of p, so (7.1) automatically preserves the Fourier-transformed incompressibility condition $\kappa_i v_i = 0$. To show this, one need only form the scalar product of (7.1) and κ_i, use the identity

$$\kappa_i\frac{\partial v_i}{\partial \kappa_j} = \frac{\partial}{\partial \kappa_j}(\kappa_i v_i) - v_j,$$

and remember that $A_{ii} = 0$ for an incompressible mean field and $\kappa_i g_i = 0$ for solenoidal body forces.

If A_{ij} and D_i are zero, then (7.1) can most easily be solved by transforming it again with respect to time. The resulting equation involves the space-time Fourier transform $\mathbf{h}(\boldsymbol{\kappa},\omega)$ of $\mathbf{f}(\mathbf{x},t)$, where

$$\mathbf{f}(\mathbf{x},t) = \int d\boldsymbol{\kappa}\int d\omega\, \mathbf{h}(\boldsymbol{\kappa},\omega)\, e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}+\omega t)}.$$

Since **f** is homogeneous and stationary, its correlation tensor

$$Q_{ij} = \langle f_i(\mathbf{x},t) f_j(\mathbf{x}+\mathbf{r}, t+s)\rangle$$

must be a function of **r** and s only, and it must also be true that

$$Q_{ij}(\mathbf{r},s) = Q_{ji}(-\mathbf{r}, -s).$$

But $$Q_{ij}(\mathbf{r},s) = \int d\boldsymbol{\kappa} \int d\boldsymbol{\kappa}' \int d\omega \int d\omega' \langle h_i(\boldsymbol{\kappa}',\omega') h_j(\boldsymbol{\kappa},\omega)\rangle \times \exp\{i(\boldsymbol{\kappa}+\boldsymbol{\kappa}').\mathbf{x} + i(\omega+\omega')t + i(\boldsymbol{\kappa}.\mathbf{r}+\omega s)\}.$$

It follows that

$$\langle h_i(\boldsymbol{\kappa}',\omega') h_j(\boldsymbol{\kappa},\omega)\rangle = \Gamma_{ij}(\boldsymbol{\kappa},\omega)\,\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}')\,\delta(\omega+\omega'), \tag{7.2}$$

where $\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}')$ is a three-dimensional delta function, $\delta(\omega+\omega')$ is a one-dimensional delta function, and $\Gamma_{ij}(\boldsymbol{\kappa},\omega)$ is a wave-number-frequency spectrum tensor satisfying

$$\Gamma_{ji}(-\boldsymbol{\kappa}, -\omega) = \Gamma_{ij}(\boldsymbol{\kappa},\omega). \tag{7.3}$$

If **f** is solenoidal and $Q_{ij}(\mathbf{r},s)$ is isotropic *for all time separations s*, then

$$\Gamma_{ij}(\boldsymbol{\kappa},\omega) = \frac{G(\kappa,\omega)}{8\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right) \tag{7.4}$$

by analogy with (3.4). The quantity κ is the positive magnitude of $\boldsymbol{\kappa}$. The frequency ω may take on either sign, but the symmetry condition (7.3) implies that

$$G(\kappa, -\omega) = G(\kappa,\omega). \tag{7.5}$$

The spectrum function G has been normalized so that

$$\frac{\langle \mathbf{f}^2\rangle}{2} = \int_0^\infty \int_0^\infty G(\kappa,\omega)\, d\kappa\, d\omega.$$

The condition that $Q_{ij}(\mathbf{r},s)$ is isotropic for every value of the time delay s is more restrictive in principle than the classical isotropy condition that no direction of a random field is distinguished at a particular time. Isotropy of $Q_{ij}(\mathbf{r},s)$ follows if no direction can be distinguished given the whole history of the random field **f**; if, at each frequency, no wave-number direction is preferred. If **f** were not isotropic in this more restrictive sense, then the force field would have a statistically discernible drift velocity. Elongated contours of $\Gamma_{ij}(\boldsymbol{\kappa},\omega)$ would distinguish a preferred frequency $\omega(\boldsymbol{\kappa})$ for each wave-number $\boldsymbol{\kappa}$, and eddy formations characterized by that wave-number would tend to drift at a velocity $d\omega/d\kappa_i$.

If A_{ij} and D_i are zero, then the Fourier time transform of equation (7.1) is

$$(i\omega + \nu\kappa^2)\, w_i = h_i,$$

where $\mathbf{w}(\boldsymbol{\kappa},\omega)$ is the time transform of $\mathbf{v}(\boldsymbol{\kappa},t)$, in other words, the space-time transform of the velocity $\mathbf{u}(\mathbf{x},t)$. According to (7.2) and (7.4),

$$\langle w_i(\boldsymbol{\kappa}',\omega')\, w_j(\boldsymbol{\kappa},\omega)\rangle = \Phi_{ij}(\boldsymbol{\kappa},\omega)\,\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}')\,\delta(\omega+\omega'), \tag{7.6}$$

for a wave-number-frequency spectrum tensor

$$\Phi_{ij} = \frac{E(\kappa,\omega)}{8\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right), \tag{7.7}$$

where $$G(\kappa,\omega) = [(\nu\kappa^2)^2 + \omega^2]\, E(\kappa,\omega). \tag{7.8}$$

Equation (7.5) implies that

$$E(\kappa, -\omega) = E(\kappa, \omega). \tag{7.9}$$

Thus a force field whose spectrum is given by (7.8) maintains weak turbulence with a spectrum $E(\kappa, \omega)$, and the turbulence is isotropic at all frequencies. The spectrum function $E(\kappa, \omega)$ is normalized so that

$$T = \int_0^\infty \int_0^\infty E(\kappa, \omega)\, d\kappa\, d\omega,$$

T being the turbulent energy density.

8. Translation stress

Equation (7.1) simplifies to

$$\frac{\partial v_i}{\partial t} + \nu\kappa^2 v_i = e^{i\boldsymbol{\kappa}.\mathbf{D}(t)} g_i,$$

if the turbulence is convected without deformation through its environment of body forces. The solution is

$$v_i(\boldsymbol{\kappa}, t) = \int_0^\infty \exp\{i\boldsymbol{\kappa}.\mathbf{D}(t-s) - \nu\kappa^2 s\}\, g_i(\boldsymbol{\kappa}, t-s)\, ds$$

for arbitrary initial conditions in the distant past. Thus

$$u_i(\boldsymbol{\xi}, t) = \int d\boldsymbol{\kappa} \int d\omega \int_0^\infty ds\, h_i(\boldsymbol{\kappa}, \omega) \exp\{i\boldsymbol{\kappa}.[\mathbf{D}(t-s) + \boldsymbol{\xi}] + i\omega(t-s) - \nu\kappa^2 s\},$$

where the Fourier time transform $\mathbf{h}(\boldsymbol{\kappa}, \omega)$ of $\mathbf{g}(\boldsymbol{\kappa}, t)$ has been introduced. According to equations (7.2), (7.4), and the symmetry condition (7.5),

$$\langle u_i u_j \rangle = 2\int_0^\infty d\kappa \int_0^\infty d\omega \int_0^\infty ds \int_0^\infty ds' G(\kappa, \omega)\, \beta_{ij}(\kappa \Delta\mathbf{D}) \exp\{i\omega(s'-s) - \nu\kappa^2(s'+s)\}, \tag{8.1}$$

where

$$\Delta\mathbf{D} = \mathbf{D}(t-s) - \mathbf{D}(t-s'),$$

and

$$\beta_{ij}(\mathbf{a}) = \frac{1}{8\pi} \oint (\delta_{ij} - n_i n_j)\, e^{i\mathbf{n}.\mathbf{a}}\, d\Omega$$

for an arbitrary vector $\mathbf{a}$. The integral in the definition of β_{ij} is carried out over the unit sphere, and $\mathbf{n}$ is the unit vector normal to the sphere. It is easy to show that

$$\beta_{ij}(\mathbf{a}) = \left(\delta_{ij} + \frac{\partial^2}{\partial a_i \partial a_j}\right) \frac{\sin a}{2a}$$

and, in particular, $\beta_{ij}(0) = \delta_{ij}/3$.

In the absence of a mean field of deformation, $\langle u_i u_j \rangle$ depends on the displacement history of the turbulence. To the extent that $\langle u_i u_j \rangle$ departs from its equilibrium value $2T\delta_{ij}/3$, a Reynolds stress develops that a viscoelasticity model cannot explain. Suppose, for example, that $d\mathbf{D}/dt = \mathbf{D}_0 \delta(t)$, so the turbulence is abruptly displaced a vector distance $\mathbf{D}_0$ through its environment of body forces. Equations (7.8) and (8.1) imply that

$$\tau_{ij} = 4\int_0^\infty \int_0^\infty E(\kappa,\omega)\, e^{-\nu\kappa^2 t}(\cos\omega t - e^{-\nu\kappa^2 t})\,[\delta_{ij}/3 - \beta_{ij}(\kappa\mathbf{D}_0)]\,d\kappa\,d\omega \tag{8.2}$$

after $t = 0$. If the co-ordinates are chosen so that the translation occurs along the x_1-axis, then $\mathbf{D}_0 = D_0\mathbf{e}_1$, and

$$\delta_{ij}/3 - \beta_{ij}(\kappa\mathbf{D}_0) = \begin{Bmatrix} 1/3-\beta_{11} & 0 & 0 \\ 0 & 1/3-\beta_{22} & 0 \\ 0 & 0 & 1/3-\beta_{33} \end{Bmatrix},$$

where

$$\beta_{11} = \frac{\sin(\kappa D_0)}{(\kappa D_0)^3} - \frac{\cos(\kappa D_0)}{(\kappa D_0)^2},$$

$$\beta_{22} = \beta_{33} = \frac{1}{2}\left[\frac{\sin(\kappa D_0)}{(\kappa D_0)} - \beta_{11}\right].$$

The stress tensor given by (8.2) appears as a system of purely normal stresses in co-ordinates aligned with the flow and is therefore fundamentally unlike a shear stress. It is zero immediately after $t = 0$, as it must be according to the argument of § 3 that stress responds instantaneously only to strain, and it tends to zero as $t \to \infty$. It is fully isotropic and is independent of $\mathbf{D}_0$ if $D_0/l \gg 1$, since $\beta_{ij}(\kappa\mathbf{D}_0)$ then undergoes high-pitched, self-cancelling oscillations during the integration over κ. It attains a maximum value of order $T(D_0/l)^2$ if $D_0/l \ll 1$. If the field $\mathbf{f}$ were not isotropic at all frequencies, then (8.1) would contain an additional term anti-symmetric in $\kappa\Delta\mathbf{D}$, and the stress due to a small, abrupt displacement would grow linearly with D_0/l.

Displacement alone, therefore, induces a delayed-action 'translation stress' depending in a highly non-linear fashion on displacement history. Translation stress arises because a moving packet of turbulence sees a non-isotropic field of body forces. The forces appear to have a statistically discernible drift velocity that distinguishes the axis of translation. The eddies distort accordingly and sustain a Reynolds stress. For relatively slow displacements, the translation stress is of order $T(U\theta/l)^2$, $U\theta$ being the distance travelled during a relaxation time. Since U is typically of order AL,

$$(\text{translation stress}) \sim (\alpha/\epsilon)^2 T. \tag{8.3}$$

If the turbulence is convected abruptly over many correlation lengths l, then the correlation $\langle \mathbf{u}.\mathbf{f}\rangle$ maintaining the turbulence in energy-equilibrium suddenly is broken, and the turbulence decays until $\langle \mathbf{u}.\mathbf{f}\rangle$ builds up enough to exceed the rate of energy dissipation. That is the meaning of the isotropic limit of (8.2) for $D_0/l \to \infty$. A viscoelasticity model is appropriate, in the presence of body forces, only when strain organizes the eddies more effectively than translation. That happens (provided $\mathbf{f}$ is isotropic) when the mean field is sufficiently weak, since translation stress then depends quadratically on the rate of displacement, but shear stress depends linearly on the rate of strain.

9. Shear stress

It is now necessary to deal with equation (7.1) for a small but otherwise general deformation rate $A_{ij}(t)$. In certain special cases involving time-independent A_{ij}, it is possible to treat (7.1) as a wave equation in $\boldsymbol{\kappa}, t$ space and solve it analytically by the method of characteristics [cf. Crow (1967*b*) and Moffatt (1965), where an approach equivalent to the method of characteristics is used to deal with the case $A_{12} \neq 0$]. In general, however, the complicated geometry of the characteristic curves excludes the possibility of a closed-form solution. But none is needed. If the exact solution for Reynolds stress were known, it would have to be linearized on A_{ij} anyway to resemble a viscoelastic constitutive law. It is sufficient to write $\mathbf{v}$ as a perturbation series,

$$\mathbf{v} = \mathbf{v}^0 + \mathbf{v}^A + \ldots,$$

comprising terms of successively higher order in α. In particular, $\mathbf{v}^A \sim O(\alpha v^0)$. The terms satisfy a hierarchy of equations obtained from (7.1):

$$\frac{\partial v_i^0}{\partial t} + \nu\kappa^2 v_i^0 = e^{i\boldsymbol{\kappa}.\mathbf{D}(t)} g_i,$$

$$\frac{\partial v_i^A}{\partial t} + \nu\kappa^2 v_i^A = A_{mn}(t)\; \mathscr{K}_{ijmn}\{v_j^0(\boldsymbol{\kappa}, t)\}, \tag{9.1}$$

and so on, where $\mathscr{K}_{ijmn}$ denotes the operator

$$\left(\frac{2\kappa_i\kappa_m}{\kappa^2}\delta_{jn} - \delta_{im}\delta_{jn} + \delta_{ij}\kappa_m\frac{\partial}{\partial\kappa_n}\right).$$

The Reynolds stress likewise can be expressed as a series:

$$\tau_{ij} = -\langle u_i^A u_j^0 + u_i^0 u_j^A \rangle + \ldots, \tag{9.2}$$

where $\mathbf{u}^0$ and $\mathbf{u}^A$ are the inverse transforms of $\mathbf{v}^0$ and $\mathbf{v}^A$. The first term in the series (9.2) leads to a linear constitutive law of the form (4.3). Higher-order terms presumably correspond to non-Newtonian effects partially represented in (4.4). Let us ignore those terms and, in order to isolate the effect of shear, assume that $\mathbf{D}$ changes very slowly. A slowly changing phase $\boldsymbol{\kappa}.\mathbf{D}$ is dynamically irrelevant, so $\mathbf{v}^0$ is the Fourier space transform of a field $\mathbf{u}^0$ of turbulence in equilibrium. The statistical relationship between $\mathbf{u}^0$ and $\mathbf{f}$ was discussed in §7. Since $\mathbf{u}^A$ depends on the interaction between A_{ij} and $\mathbf{u}^0$ [cf. equation (9.1)] but does not depend explicitly on $\mathbf{f}$, no further mention of the body forces need be made. The Reynolds stress given by (9.1) and (9.2) depends only on the deformation history of the turbulence and on its equilibrium structure.

The solution of (9.1) is

$$v_i^A(\boldsymbol{\kappa}, t) = \int_0^\infty e^{-\nu\kappa^2 s} A_{mn}(t-s)\; \mathscr{K}_{ijmn}\{v_j^0(\boldsymbol{\kappa}, t-s)\}\, ds$$

for arbitrary conditions in the distant past. Thus

$$u_i^A(\boldsymbol{\xi}, t) = \int d\boldsymbol{\kappa} \int d\omega \int_0^\infty ds \exp\{-\nu\kappa^2 s + i\omega(t-s) + i\boldsymbol{\kappa}.\boldsymbol{\xi}\} \times A_{mn}(t-s)\, \mathscr{K}_{ijmn}\{w_j^0(\boldsymbol{\kappa}, \omega)\},$$

where $\mathbf{v}^0$ has been expressed in terms of its Fourier time transform $\mathbf{w}^0$. Integration by parts over wave-number space yields

$$u_i^A(\boldsymbol{\xi}, t) = \int d\boldsymbol{\kappa} \int d\omega \int_0^\infty ds \exp\{-\nu\kappa^2 s + i\omega(t-s) + i\boldsymbol{\kappa}.\boldsymbol{\xi}\} A_{mn}(t-s) K_{ijmn} w_j^0(\boldsymbol{\kappa}, \omega), \tag{9.3}$$

where K_{ijmn} is the algebraic multiplier

$$\left(\frac{2\kappa_i\kappa_m}{\kappa^2}\delta_{jn} - \delta_{im}\delta_{jn} - \delta_{ij}\delta_{mn} + 2\nu\kappa_m\kappa_n s\delta_{ij} - i\delta_{ij}\kappa_m\xi_n\right).$$

By definition,

$$u_j^0(\boldsymbol{\xi}, t) = \int d\boldsymbol{\kappa}' \int d\omega' w_j^0(\boldsymbol{\kappa}', \omega') \exp\{i\boldsymbol{\kappa}'.\boldsymbol{\xi} + i\omega' t\}.$$

Therefore

$$\langle u_i^A(\boldsymbol{\xi}, t)\, u_j^0(\boldsymbol{\xi}, t)\rangle = -\frac{2}{15}\int_0^\infty d\kappa \int_0^\infty d\omega \int_0^\infty ds E(\kappa, \omega) \times (1+\nu\kappa^2 s)\, e^{-\nu\kappa^2 s} \cos\omega s[A_{ij}(t-s) + A_{ji}(t-s)], \tag{9.4}$$

according to the incompressibility condition (2.2), the geometrical relations (3.5), the homogeneity and isotropy conditions (7.6) and (7.7), and the symmetry condition (7.9). The symmetry condition plays a central role. If the spectrum tensor $\Phi_{ij}(\boldsymbol{\kappa}, \omega)$ were not isotropic at each frequency ω, then (9.4) would contain an additional term, linear in $\boldsymbol{\xi}$, representing translation stress generated in the outer regions of the packet, whose centre, by assumption, is moving only slowly. If a preferred direction could be distinguished by watching the equilibrium turbulence over a period of time, then the effects of translation and shear would be inseparable.

Equations (9.2) and (9.4) yield a constitutive law of the form

$$\tau_{ij} = \tfrac{4}{15} T \int_0^\infty \mathfrak{M}(s)\,[A_{ij}(t-s) + A_{ji}(t-s)]\, ds,$$

which is equivalent to (4.3), since the analysis has been carried out in co-ordinates moving with the packet of turbulence. The memory function is determined explicitly in terms of the wave-number-frequency spectrum of the background turbulence:

$$\mathfrak{M}(s) = \int_0^\infty \int_0^\infty \frac{E(\kappa, \omega)}{T}\,(1+\nu\kappa^2 s)\, e^{-\nu\kappa^2 s} \cos\omega s\, d\kappa\, d\omega. \tag{9.5}$$

The integral relaxation time θ is found by integrating $\mathfrak{M}(s)$ over s:

$$\theta = \int_0^\infty \int_0^\infty \frac{E(\kappa, \omega)}{T} \frac{2(\nu\kappa^2)^3}{[\omega^2 + (\nu\kappa^2)^2]^2} d\kappa\, d\omega. \tag{9.6}$$

The integrand of (9.5) contains the factor $(1+\nu\kappa^2 s)\exp(-\nu\kappa^2 s)$, representing stress relaxation by viscous diffusion, and the factor $\cos\omega s$, representing relaxation by random agitation from the body forces. If the contribution of molecular diffusion is small, then

$$\mathfrak{M}(s) \approx \int_0^\infty \int_0^\infty \frac{E(\kappa, \omega)}{T} \cos\omega s\, d\kappa\, d\omega = \frac{\langle \mathbf{u}(\mathbf{x}, t).\mathbf{u}(\mathbf{x}, t+s)\rangle}{\langle \mathbf{u}^2\rangle}$$

for an equilibrium field $\mathbf{u}$. In that case, $\mathfrak{M}(s)$ is just the Eulerian time correlation of $\mathbf{u}$, a result that makes qualitative sense even for strong turbulence.

These results show that shear stress does indeed evolve according to a linear viscoelastic constitutive law, provided that the turbulence is weak, that ϵ and α are small, and that the turbulent packet in question translates slowly enough through its sustaining environment of body forces. 'Slowly enough' means that translation stress is locally much smaller than shear stress. But

$$\text{(shear stress)} \sim \alpha T. \tag{9.7}$$

Judged from the order-of-magnitude estimates (8.3) and (9.7), translation stress is negligible compared with shear stress everywhere in the flow (rather than merely at isolated points where $d\mathbf{D}/dt$ happens to be especially small) only if the mean field is so weak that $\alpha \ll \epsilon^2$.

10. Concluding remarks

The results of §9 show that weak turbulence maintained by body forces can behave viscoelastically, but only under three conditions. First, the turbulence must be fine-grained, $\epsilon \ll 1$, so that second- and higher-order derivatives of the mean field can be disregarded in equations (6.1) and (6.2). Second, the rate of deformation must be small, $\alpha \ll 1$, so that equation (7.1) can be solved by perturbation methods and higher-order terms in the series (9.2) for Reynolds stress can be neglected. Third, the mean field must be sufficiently weak that the effect of translation on the eddy structure is negligible, since otherwise translation stress dominates shear stress. The third condition is satisfied everywhere in the flow only if $\alpha \ll \epsilon^2$, a severe inequality that overrides the second condition. The viscoelastic constitutive law (4.3) was originally derived by combining a general result for the initial elastic response of turbulence with qualitative arguments about the subsequent relaxation of Reynolds stress, arguments independent of the actual relaxation mechanisms. The arguments appeared to be justified if only $\epsilon, \alpha \ll 1$. The work of §§6–9 casts those arguments into quantitative form, for turbulence so weak that only the linear relaxation processes of molecular diffusion and body-force agitation need be considered, and shows that the arguments are fully justified only if $\epsilon \ll 1$ and $\alpha \ll \epsilon^2$. The surprisingly severe restriction on α is necessary to eliminate translation stress, a delayed-action non-linear response that could not have been anticipated from the initial elastic response of turbulence. It is reasonable to assume that the restrictions $\epsilon \ll 1$, $\alpha \ll \epsilon^2$ apply also to the viscoelastic behaviour of strong turbulence whenever the presence of body forces must be taken seriously. In a mean field of vanishing strength, the Reynolds stress would evolve according to the constitutive law (4.3), with a memory function $\mathfrak{M}(t)$ depending primarily on relaxation by non-linear scrambling. In practice, however, α rarely would be much less than ϵ^2, and the effect of forcing the turbulence through a field of body forces would dominate the effect of straining it.

What saves (4.3) from being an academic curiosity is that it may provide a reasonably accurate description of a body of turbulence that is not quite homogeneous and isotropic, but is driven by means more realistic than body forces, by a flux of random vorticity from the vicinity of a solid boundary, for example. Translation stress is strictly an artifact of body forces, not an intrinsic non-linearity of the kind represented in (4.4). If the body forces can be eliminated, then the condition $\alpha \ll \epsilon^2$ can be relaxed back to $\alpha \ll 1$, or perhaps even to $\alpha \lesssim 1$ with the aid of a non-linear viscoelastic constitutive law of the type (4.4). Nowhere in the course of the analysis of shear stress in §9 did the body forces appear explicitly. Equation (9.1) is valid if $\alpha \ll 1$ and equation (9.3) follows directly from it regardless of what is assumed about the background field $\mathbf{u}^0$. Body forces intervene between (9.3) and (9.4) only in the sense that they guarantee a rigorously homogeneous and isotropic $\mathbf{u}^0$. If $\mathbf{u}^0$ can be supposed sufficiently homogeneous and isotropic on other grounds, then (9.4) follows anyway [more correctly, the analogue of (9.4) for strong turbulence follows anyway, since there are no other grounds than body forces for assuming weak turbulence to be homogeneous and isotropic; only in strong turbulence can random vorticity propagate by self-induction away from a boundary].

Equation (3.7), which gives the initial response of turbulence of any strength to an arbitary small deformation, and (9.5), which describes the relaxation of Reynolds stress in weak turbulence, are the main quantitative results derived in this paper. It remains to consider whether the memory function $\mathfrak{M}(t)$ can be predicted for strong turbulence. An expression analogous to (9.5) is needed, relating $\mathfrak{M}(t)$ to some relatively simple properties of the background turbulence. The expression might have to incorporate more complicated statistical properties than the wave-number-frequency spectrum $E(\kappa, \omega)$, but it may be that turbulence attains equilibrium states determined more-or-less uniquely by the spectrum, in which case some non-linear functional of $E(\kappa, \omega)$ alone would appear in the analogue of (9.5). Some of the quasi-analytical theories of decaying homogeneous turbulence probably could be modified to predict $\mathfrak{M}(t)$ in stationary turbulence. The memory function $\mathfrak{M}(t)$ appears to be closely related to Kraichnan's impulse response function, for example. Predicting $\mathfrak{M}(t)$ should be just as difficult as predicting the energy content of freely decaying turbulence, however, so for the time being $\mathfrak{M}(t)$ is best regarded as a scalar memory function to be found by a suitable relaxation test on the basis of (4.1). The major conceptual advantage of a viscoelasticity theory of turbulent shear flow is that it enables us to separate the aspect of Reynolds stress evolution that can be treated analytically, initial elastic response, from the aspect that as yet cannot, return to isotropy under non-linear scrambling.

This work was performed under the auspices of the United States Atomic Energy Commission.

REFERENCES

Batchelor, G. K. 1953 *The Theory of Homogeneous Turbulence.* Cambridge University Press.

Clauser, F. H. 1956 The turbulent boundary layer. *Adv. Appl. Math.* **4,** 1.

Crow, S. C. 1967*a* Visco-elastic character of fine-grained isotropic turbulence. *Phys. Fluids,* **10,** 1587.

Crow, S. C. 1967*b* Viscoelastic properties of fine-grained Burgers turbulence. *Univ. Calif. Rad. Lab. Rept.* no. UCRL-70354.

LIEPMANN, H. W. 1961 Free turbulent flows. *Mecanique de la Turbulence.* Paris: CNRS.

MILES, J. W. 1967 On the generation of surface waves by shear flows. Part 5. *J. Fluid Mech.* **30**, 163.

MOFFATT, H. K. 1965 The interaction of turbulence with rapid uniform shear. *Stanford University SUDAER Rept.* no. 242.

PEARSON, J. R. A. 1959 The effect of uniform distortion on weak homogeneous turbulence. *J. Fluid Mech.* **5**, 274.

REYONLDS, O. 1894 On the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Phil. Trans. Roy. Soc.* A **186**, 123.

RIVLIN, R. S. 1957 The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids. *Q. appl. Math.* **15**, 212.

TAYLOR, G. I. 1935 Turbulence in a contracting stream. *Z. Angew. Math. Mech.* **15**, 91.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow.* Cambridge University Press.

TOWNSEND, A. A. 1966 The mechanism of entrainment in free turbulent flows. *J. Fluid Mech.* **26**, 689.